International Journal on Emerging Technologies 13(2): 06-11(2022)

ISSN No. (Print): 0975-8364
ISSN No. (Online): 2249-3255

Mathematical Modelling of Prey and Predator Species with Two Age Group of Prey Population

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(Received 20 June 2022, Revised 26 July 2022, Accepted 10 August 2022)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: In this paper we developed mathematical model of prey and predator wild life population of finite size. We consider a prey population can be divided into two age groups. One part of prey species are adults and other part of prey species are juveniles or infants. A mathematical framework has been generated in terms of a system of nonlinear difference equations including all the significant parameters. The solutions are worked out in terms of finite polynomial. Numerical computations have been made for the change of populations in successive generations and different values of the parameters. Graphs have been plotted to depict the comparative variations.

Keywords: Migration, Juveniles, nth generation, mathematical.

I. INTRODUCTION

Mathematical Modelling of Ecological Systems is emerging as an important area of Mathematics. Efforts to Mathematical Model such ecosystems have been done in diverse settings like Viruses, Microorganisms, wild life and demography.

Dynamics of Single Species and Multispecies populations living in a confined geographical area has always attracted the attention of mathematicians [13]. These studies differ from one population species to another depending upon the type of growth and environment involved. Evolution of any biological population is governed by multiple parameters like Growth /Reproduction, Migration and Decay/Death. Whereas isolated single species might grow independently with each unit dividing itself, multispecies populations grow through their interaction with each other. The mathematical behaviour of any such system can be discrete, continuous, piecewise continuous and transitional.

There have ingenious efforts to comprehend the evolution of such systems through Mathematical Models. The attributes of a good mathematical model for ecological setting are described by Gilpin and Ayala (1973) [6, 13]. An elaborate account of the mathematical models in ecology [8, 10].

The initial attempts to apply mathematical tools to predict the evolution of the biological systems can be traced back to Malthus (1798) which was mathematically formalized by Verhulst (1838) [4, 5]. The cyclic variations of the prey and predator populations have been dealt in the pioneering work of Lotaka(1925) and Voltera (1926) [9, 18] which was extensively discussed by Elton (1948) ; Hutchinson (1948) [3, 7]. It has been a matter of intense research activity since then. The mathematical models of single and two interacting species with special reference to protected wild life have been extensively reviewed by Chaturvedi (2012) [1]. Earlier mathematical models have largely assumed that entire population participates in reproduction. Such an assumption can make sense only for the species with short life spans and little infancy period. But in most of the animal species fertility starts after a considerable prematurity period and remains consistent for rest of the life. Similarly, there are animals with three significant age groups of pre-fertile, fertile and non fertile age group. Hence, a realistic treatment of the prey predator dynamics requires the age group structure to be incorporated in the mathematical treatment. Efforts have been made to incorporate such age structure by dividing the population in productive and non-productive age groups [2, 14, 17].

We have presented here a Mathematical Model with two interacting species which can have applications over a variety of ecological settings. The model is presented here based on a simplified picture of the evolution of prey predator dynamics. Following assumptions have been made

1. Birth rate is constant throughout the generations
2. The food supply for the predator population depends exclusively on the size of the prey population.
3. Rate of change of the prey's population depends entirely upon its own growth rate minus the rate at which it is preyed upon and the natural death and hunting of prey population induced by its own biological evolution has been ignored. However, the evolution of predator's population has significant bearing upon the rate at which it consumes prey, minus its intrinsic death rate.
4. During the course of evolution, the environment does not change in favour of one species and there is no genetic mutation which can significantly affect the population dynamics of prey and predator population.

II. FORMULATION OF THE PROBLEM

We assume that P_n denotes the prey population and it is divided into two parts as given below

$$P_n = {P_n}^{(1)} + {P_n}^{(2)}$$

Where ${P_n}^{(1)} =$ Prey population of Juveniles
${P_n}^{(2)} =$ Prey population of fertile age group

The change in juvenile prey population in nth generation is given by the equation

$$\Delta {P_n}^{(1)} = B{P_{n-1}}^{(2)} - D_n{P_{n-1}}^{(1)} - T_n{P_{n-1}}^{(1)} - \alpha_n Q{P_{n-1}}^{(1)} + {M_n}^{(1)}$$

Where,
$B =$ Birth rate of juveniles
$D_n =$ Death rate of juvenile
$T_n =$ Transition rate of juveniles
$\alpha_n =$ Killing rate of juveniles by Predators
${M_n}^{(1)} =$ Migration of juvenile

Since

$$\Delta {P_n}^{(1)} = {P_n}^{(1)} - {P_{n-1}}^{(1)}$$

Therefore, the change in juvenile prey population in nth generation is given by the equation

$$P_n^{(1)} = (1 - D_n - T_n - \alpha_n Q)P_{n-1}^{(1)} + BP_{n-1}^{(2)} + M_n^{(1)}$$

Let

$x_n = \quad 1 - D_n - T_n - \alpha_n Q$

Then

$$P_n^{(1)} = x_n P_{n-1}^{(1)} + BP_{n-1}^{(2)} + M_n^{(1)} \qquad (1)$$

Similarly

The change in fertile prey population in nth generation is given by the equation

$$\Delta P_n^{(2)} = T_n P_{n-1}^{(1)} - D_n P_{n-1}^{(2)} - \alpha_n Q P_{n-1}^{(2)} + M_n^{(2)}$$

Where,

D_n = Death rate of fertile age group population

T_n = Transition rate of juveniles from $P_{n-1}^{(1)}$ to $P_{n-1}^{(2)}$

α_n = Killing rate of fertile age group population by Predators

$M_n^{(2)}$ = Migration of fertile age group population

Let

$\mu$is prescribed ratio and the population of the juveniles age group is less than the fertile age group. This mean that μ is strictly less than 1.

Then

$$P_n^{(1)} = \mu P_n^{(2)}, (\mu < 1)$$

Since

$$\Delta P_n^{(2)} = P_n^{(2)} - P_{n-1}^{(2)}$$

Therefore, $P_n^{(2)} = (1 + T_n\mu - D_n - \alpha_n Q)P_{n-1}^{(2)} + M_n^{(2)}$

Let

$y_n = \quad 1 + T_n\mu - D_n - \alpha_n Q$

Then

$$P_n^{(2)} = y_n P_{n-1}^{(2)} + M_n^{(2)} \qquad (2)$$

The Solution of this difference equations is given below

$$P_1^{(2)} = y_1 P_0^{(2)} + M_1^{(2)}$$
$$P_2^{(2)} = y_2 P_1^{(2)} + M_2^{(2)}$$

or

$$P_2^{(2)} = y_2 y_1 P_0^{(2)} + y_2 M_1^{(2)} + M_2^{(2)}$$

Thus, by induction we get

$$P_n^{(2)} = y_n y_{n-1} y_{n-2} \dots\dots\dots y_4 y_3 y_2 y_1 P_0^{(2)} + \cdots \dots\dots + y_n y_{n-1} y_{n-2} \dots\dots\dots y_4 y_3 y_2 M_1^{(2)} + y_n y_{n-1} y_{n-2} \dots\dots\dots y_4 y_3 M_2^{(2)} + \cdots \dots + y_n y_{n-1} y_{n-2} M_{n-3}^{(2)} + y_n y_{n-1} M_{n-2}^{(2)} + y_n M_{n-1}^{(2)} + M_n^{(2)}$$

Or

$$P_n^{(2)} = \prod_{i=1}^{n} y_i P_0^{(2)} + \sum_{j=1}^{n-1} \prod_{i=j+1}^{n} y_i M_j^{(2)} + M_n^{(2)} \qquad (3)$$

Replace n by n-1, we get

$$P_{n-1}^{(2)} = \prod_{i=1}^{n-1} y_i P_0^{(2)} + \sum_{j=1}^{n-2} \prod_{i=j+1}^{n-1} y_i M_j^{(2)} + M_{n-1}^{(2)}$$

Putting this value in eq. (1), we get

$$P_n^{(1)} = x_n P_{n-1}^{(1)} + B[\prod_{i=1}^{n-1} y_i P_0^{(2)} + \sum_{j=1}^{n-2} \prod_{i=j+1}^{n-1} y_i M_j^{(2)} + M_{n-1}^{(2)}] + M_n^{(1)}$$

The solution of this difference equation is given by

$$P_1^{(1)} = x_1 P_0^{(1)} + BP_0^{(2)} + M_1^{(1)}$$
$$P_2^{(1)} = x_2 P_1^{(1)} + B(y_1 P_0^{(2)} + M_1^{(2)}) + M_2^{(1)}$$

Or

$$P_2^{(1)} = x_2(x_1 P_0^{(1)} + BP_0^{(2)} + M_1^{(1)}) + B(y_1 P_0^{(2)} + M_1^{(2)}) + M_2^{(1)}$$

Or

$$P_2^{(1)} = x_2 x_1 P_0^{(1)} + B(x_2 + y_1)P_0^{(2)} + x_2 M_1^{(1)} + BM_1^{(2)} + M_2^{(1)}$$

Finally, we obtain

$$P_n^{(1)} = \prod_{i=1}^{n} x_i P_0^{(1)} + B\sum_{r=1}^{n} \prod_{i=r+1}^{n} x_i \prod_{j=1}^{r-1} y_j P_0^{(2)} + \sum_{r=1}^{n} \prod_{i=r+1}^{n} x_i M_r^{(1)} + B\sum_{m=1}^{n-1} \sum_{r=m+1}^{n} \prod_{i=r+1}^{n} x_i \prod_{j=m+1}^{r-1} y_j M_m^{(2)} \qquad (4)$$

Special Case:

Putting

$y_1 = y_2 = y_3 = y_4 = y_5 \dots\dots\dots\dots\dots\dots\dots y_n$ = y in eq. (3)

$$P_n^{(2)} = y^n P_0^{(2)} + \sum_{r=1}^{n} M_r^{(2)} y^{n-r}$$

Or

$$P_n^{(2)} = y^n P_0^{(2)} + \sum_{r=0}^{n-r} M_{n-r}^{(2)} y^r$$

After replacing n by n', we get

$$P_{n'}^{(2)} = y^{n'} P_0^{(2)} + \sum_{r=0}^{n'-r} M_{n'-r}^{(2)} y^r \qquad (5)$$

Right hand side is a polynomial in y. We take M's as the coefficient of hypergeometric series of the form [1]

$$2F_1[-n,b;c;y] = \sum_{r=0}^{n} \frac{(-n)_r (b)_r y^r}{(c)_r r!} \qquad (6)$$

We consider the case when y< 1.

Now Let

$$\sum_{r=0}^{n'-r} M_{n'-r}^{(2)} y^r = \sum_{r=0}^{n} \frac{(-n)_r (b)_r y^r}{(c)_r r!} \qquad (7)$$

and n'=n+1.

Then after comparing the coefficients we get

$$M_{n'-r}^{(2)} = \frac{(-n)_r (b)_r}{(c)_r r!} \qquad (8)$$

Putting the value in eq. (5), we get

$$P_{n+1}^{(2)} = y^{n+1} P_0^{(2)} + \sum_{r=0}^{n} \frac{(-n)_r (b)_r y^r}{(c)_r r!}$$

Or

$$P_{n-1}^{(2)} = y^{n-1} P_0^{(2)} + \sum_{r=0}^{n-2} \frac{(-n+2)_r (b)_r y^r}{(c)_r r!} \qquad (9)$$

Putting the value in eq. (1), we get

$$P_n^{(1)} = x_n P_{n-1}^{(1)} + B[y^{n-1} P_0^{(2)} + \sum_{r=0}^{n-2} \frac{(-n+2)_r (b)_r y^r}{(c)_r r!}] + M_n^{(1)}$$

Step – I (n=1)

$$P_1^{(1)} = x_1 P_0^{(1)} + BP_0^{(2)} + M_1^{(1)}$$

Step - II (n=2)

$$P_2^{(1)} = x_2 x_1 P_0^{(1)} + B(x_2 + y_1)P_0^{(2)} + x_2 M_1^{(1)} + BM_1^{(2)} + M_2^{(1)}$$

If $\quad x_2 = x_1 = $ x and$y_1 = y, \quad M_1^{(1)} = M_2^{(1)}$= M $\; T_n = T, D_n = D$

Then

$P_2^{(1)} = x^2 P_0^{(1)}$+ B(x+y)$P_0^{(2)}$ + (x +1) M + B $M_1^{(2)}$

Let

$$P_{n-1}^{(1)} = \mu P_{n-1}^{(2)}$$

Since μ is prescribed ratio and 0< $\mu < 1$

Then

$$P_n^{(2)} = y_n P_{n-1}^{(2)} + T_n \mu P_{n-1}^{(2)} + M_n^{(2)}$$

Solution of Difference Equations:

$$P_1^{(1)} = x_1 P_0^{(1)} + BP_0^{(2)} + M_0^{(1)} \qquad (10)$$
$$P_1^{(2)} = y_1 P_0^{(2)} + T_1 P_0^{(1)} + M_1^{(2)} \qquad (11)$$
$$P_2^{(1)} = x_2 P_1^{(1)} + BP_1^{(2)} + M_1^{(1)}$$
$$P_2^{(1)} = (x_2 x_1 P_0^{(1)}) + B(x_2 P_0^{(2)} + y_1 P_0^{(2)} + T_1 P_0^{(1)} + M_1^{(2)}) + x_2 M_0^{(1)} + M_1^{(1)}$$

Hence we can contextualize the mathematical treatment as given by Saxena (2011) for Prey dynamics as well. The new terms which appear in these equations in place of migration have altogether different physical significance. However, their mathematical evolution will be governed by similar mathematical structures [14].

Let

$$1+\ \alpha_n-\beta_n Q_{n-1}{}^{(2)}=\ g_n$$

Then

$$P_n=g_n P_{n-1}-\ \ M_n \qquad (12)$$

The solution of the difference equation (12) can be given by

$$P_n=g_n g_{n-1} g_{n-2}\dots\dots\dots .g_3 g_2 g_1 P_0 - g_n g_{n-1} g_{n-2}\dots\dots\dots .g_4 g_3 g_2 M_1 - g_n g_{n-1} g_{n-2}\dots\dots\dots\dots .g_5 g_4 g_3 M_2 - \dots\dots\dots\dots - g_n g_{n-1} g_{n-2} M_{n-3} - g_n g_{n-1} M_{n-2}-\ g_n M_{n-1}-\ M_n$$

Or

$$P_n=\prod_{i=1}^{n} g_i P_0-\left(\sum_{j=1}^{n-1}\prod_{i=j+1}^{n} g_i M_j\right)-M_n \qquad (13)$$

If $g_1=g_2=g_3=g_4=\ \dots\dots\dots\dots\dots\dots\dots .g_n=\ \ g$, then

$$P_n=g^n P_0-\left(M_n+gM_{n-1}+g^2M_{n-2}+g^3M_{n-3}+\dots\dots\dots\dots\dots\dots\dots+g^{n-1}M_1\right)$$

$$P_n=g^n P_0-\sum_{r=0}^{n-1} M_{n-r}\, g^r \qquad (14)$$

Right hand side is a polynomial in g.

Solution in Terms of Classical Polynomials

Relation (14) for specific cases of killing pattern can be expressed in terms of classical polynomials by manipulating the coefficient of (g) keeping in view the desired pattern of growth /decay of the population in confined land or water body. The dynamic pattern can be related with available resources like water, food, vegetation and natural shelter. Some specific polynomial applications are given below

(i) Hermite Polynomials

$$H_n(x)=\ \sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$$

Assuming that the desired growth will follow the pattern of this polynomial. Accordingly from the equation (14) we get,

$$\sum_{r=0}^{[n-1]} M_{n-r}\, g^r=\ \sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$$

Where n= $\frac{n}{2}+1$

After comparing the coefficients, we get

$$M_{n-r}=\ \frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!} \qquad (15)$$

(ii) Laguerre Polynomials:

$$L_n(x)=\sum_{r=0}^{[n]}\frac{(-1)^r n!(x)^r}{(n-r)!(r!)^2}$$

Comparing with equation (14) we get,

$$\sum_{r=0}^{[n-1]} M_{n-r}\, g^r=\ \sum_{r=0}^{[n]}\frac{(-1)^r n!}{(n-r)!(r!)^2}\, g^r$$

Where n= n+1

After comparing the coefficients, we get

$$M_{n-r}=\frac{(-1)^r n!}{(n-r)!(r!)^2} \qquad (16)$$

(iii) Jacobi Polynomials:

$$P_n^{(\alpha,\delta)}(1-\alpha)=\sum_{k=0}^{n}\frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n\alpha^k}{n!(1+\gamma)_k 2^k}$$

We consider the case when $\alpha<1$ i.e. when per capita birth rate is smaller than the killing rate.

From equation (14) we get,

$$\sum_{r=0}^{[n-1]} M_{n-r}\, g^r=\ \sum_{k=0}^{n}\frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n\alpha^k}{n!(1+\gamma)_k 2^k}$$

Where n = n+1

After comparing the coefficients, we get

$$M_{n-r}=\ \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n}{n!(1+\gamma)_k 2^k} \qquad (17)$$

(iv) Gauss Hyper-Geometric Function

$$2F_1[-n,b;c;y]=\ \sum_{r=0}^{n}\frac{(-n)_r(b)_r y^r}{(c)_r r!}$$

We consider the case when $\alpha<1$ then, from equation (14) we get,

$$\sum_{r=0}^{[n-1]} M_{n-r}\, g^r=\sum_{r=0}^{n}\frac{(-n)_r(b)_r y^r}{(c)_r r!}$$

Where n = n+1

After comparing the coefficients, we get

$$M_{n-r}=\ \frac{(-n)_r(b)_r}{(c)_r r!} \qquad (18)$$

The change in active predator population of nth generation is given by the equation

$$\Delta Q_n{}^{(2)}=T_n{}^{(1)}Q_{n-1}{}^{(1)}-\delta_n Q_{n-1}{}^{(2)}-\ N_n \qquad (19)$$

Where,

$T_n{}^{(1)}$ = Transition rate of $Q_{n-1}{}^{(1)}$ to $Q_{n-1}{}^{(2)}$

δ_n = Death rate of $Q_{n-1}{}^{(2)}$

N_n =Hunting rate of $Q_{n-1}{}^{(2)}$

Since

$$\Delta Q_n{}^{(2)}=Q_n{}^{(2)}-Q_{n-1}{}^{(2)}$$

Then $\quad Q_n{}^{(2)}=(1-\delta_n)Q_{n-1}{}^{(2)}+T_n{}^{(1)}Q_{n-1}{}^{(1)}-\ N_n$

Since wild life species are generally in a fixed age group proposition. So we can write,

$$Q_n{}^{(1)}=\lambda Q_n{}^{(2)}$$

Then, $\quad Q_n{}^{(2)}=(1-\delta_n+\lambda T_n{}^{(1)})Q_{n-1}{}^{(2)}-\ N_n$

Hence we can contextualize the mathematical treatment as given by Saxena (2011) for Predator dynamics as well. The new terms which appear in these equations in place of migration have altogether different physical significance. However their mathematical evolution will be governed by similar mathematical structures [14].

Let

$$1-\delta_n+\lambda T_n{}^{(1)}=a_n$$

Then,

$$Q_n{}^{(2)}=a_n Q_{n-1}{}^{(2)}-\ N_n \qquad (20)$$

The solution of the difference equation (20) can be given by

$$Q_n{}^{(2)}=\ a_n a_{n-1}a_{n-2}a_{n-3}\dots\dots\dots\dots\dots .a_3a_2a_1Q_0{}^{(2)}+a_n a_{n-1}a_{n-2}a_{n-3}\dots\dots\dots\dots\dots .a_3a_2N_1+\dots\dots\dots\dots\dots\dots\dots+\ a_n a_{n-1}a_{n-2}a_{n-3}N_{n-4}+\ a_n a_{n-1}a_{n-2}N_{n-3}+a_n a_{n-1}N_{n-2}+\ a_n N_{n-1}+\ N_n$$

Or

$$Q_n{}^{(2)}=\prod_{i=1}^{n} a_i\, Q_0{}^{(2)}+\ \sum_{j=1}^{n-1}\prod_{i=j+1}^{n} a_i N_j+N_n \qquad (21)$$

If $a_1=a_2=a_3=a_4=\ \dots\dots\dots\dots\dots\dots\dots .a_n=\ \ a$, then

$$Q_n{}^{(2)}=a^n Q_0{}^{(2)}+\quad(N_n+aN_{n-1}+a^2N_{n-2}+a^3N_{n-3}+\dots\dots\dots\dots\dots\dots+a^{n-1}N_1)$$

$$Q_n{}^{(2)}=a^n Q_0{}^{(2)}+\sum_{r=0}^{n-1} N_{n-r}a^r \qquad (22)$$

Right hand side is a polynomial in a.

Solution in terms of Classical Polynomials

Relation (22) for specific cases of killing pattern can be expressed in terms of classical polynomials by manipulating the coefficient of (α) keeping in view the desired pattern of growth /decay of the population in confined land or water body . The dynamic pattern can be related with available resources like water, food, vegetation and natural shelter. Some specific polynomial applications are given below

(i) Hermite Polynomial:

$$H_n(x)=\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^m n!\,(2x)^{n-2m}}{m!\,(n-2m)!}$$

Comparing with equation (22) we get

$$\sum_{r=0}^{[n-1]} N_{n-r}a^r=\ \sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$$

And n= $\frac{n}{2}+1$

After comparing the coefficients, we get

$$N_{n-r}=\frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!} \qquad (23)$$

(ii) Laguerre Polynomial:

$$L_n(x) = \sum_{r=0}^{[n]} \frac{(-1)^r n!\,(x)^r}{(n-r)!\,(r!)^2}$$

Comparing with equation (22) we get,

$$\sum_{r=0}^{[n-1]} N_{n-r} a^r = \sum_{r=0}^{[n]} \frac{(-1)^r n!\,(x)^r}{(n-r)!\,(r!)^2}$$

Where n= n+1

Then after comparing the coefficients, we get

$$N_{n-r} = \sum_{r=0}^{[n]} \frac{(-1)^r n!}{(n-r)!(r!)^2} \quad (24)$$

(iii) Jacobi Polynomials:

$$P_n^{(\alpha,\delta)}(1-\alpha) = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n\alpha^k}{n!\,(1+\gamma)_k 2^k}$$

We consider the case when $\alpha < 1$ i.e. when per capita birth rate is smaller than the killing rate.

From equation (22) we get

$$\sum_{r=0}^{[n-1]} N_{n-r} a^r = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n\alpha^k}{n!(1+\gamma)_k 2^k}$$

Where n= n+1

Then after comparing the coefficients, we get

$$N_{n-r} = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n}{n!(1+\gamma)_k 2^k} \quad (25)$$

(iv) Gauss Hyper Geometric Function:

$$2F_1[-n,b;c;y] = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$$

We consider the case when $\alpha < 1$then, from equation (22) we get,

$$\sum_{r=0}^{n-1} N_{n-r} a^r = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$$

Where n= n+1

Then after comparing the coefficients, we get

$$N_{n-r} = \frac{(-n)_r(b)_r}{(c)_r r!} \quad (26)$$

The change in the inactive population in the nth generation is given by

$$\Delta Q_n^{(1)} = B_n Q_{n-1}^{(2)} P_{n-1} - T_n^{(1)} Q_{n-1}^{(1)} \quad (27)$$

We assumed that the change in the active predator population depends on the two major factors:

$B_n =$ Birth rate of $Q_{n-1}^{(2)}$

$T_n =$ Transitio rate of $Q_{n-1}^{(1)}$ to $Q_{n-1}^{(2)}$

Since

$$\Delta Q_n^{(1)} = Q_n^{(1)} - Q_{n-1}^{(1)}$$

Then $Q_n^{(1)} = (1 - T_n^{(1)}) Q_{n-1}^{(1)} + B_n Q_{n-1}^{(2)} P_{n-1}$

Hence, we can contextualize the mathematical treatment as given by Saxena (2011) for non-active Predator dynamics as well. The new terms which appear in these equations in place of migration have altogether different physical significance. However, their mathematical evolution will be governed by similar mathematical structures [14].

Let

$$1 - T_n^{(1)} = z_n$$

And $B_n Q_{n-1}^{(2)} P_{n-1} = S_n$

Then

$$Q_n^{(1)} = z_n Q_{n-1}^{(1)} + S_n \quad (28)$$

The solution of the difference equation (28) can be given by

$$Q_n^{(1)} = z_n z_{n-1} z_{n-2} z_{n-3} \ldots\ldots z_3 z_2 z_1 Q_0^{(1)} + z_n z_{n-1} z_{n-2} \ldots\ldots z_4 z_3 z_2 S_1 + z_n z_{n-1} z_{n-2} \ldots\ldots z_5 z_4 z_3 S_2 + \ldots\ldots + z_n z_{n-1} z_{n-2} S_{n-3} + z_n z_{n-1} S_{n-2} + z_n S_{n-1} + S_n$$

Or

$$Q_n^{(1)} = \prod_{i=1}^{n} z_i Q_0^{(1)} + \sum_{j=1}^{n-1} \prod_{i=j+1}^{n} z_i S_j + S_n \quad (29)$$

If $z_1 = z_2 = z_3 = z_4 = \ldots\ldots z_n = z$, then

$$Q_n^{(1)} = z^n Q_0^{(1)} + (s_n + zS_{n-1} + zS_{n-2} + zS_{n-3} + \ldots\ldots + z^{n-1} S_1)$$

$$Q_n^{(1)} = z^n Q_0^{(1)} + \sum_{r=0}^{n-1} S_{n-r} z^r \quad (30)$$

Right hand side is a polynomial in z

Solution in terms of Classical Polynomials

Relation (2.30) for specific cases of killing pattern can be expressed in terms of classical polynomials by manipulating the coefficient of (z) keeping in view the desired pattern of growth /decay of the population in confined land or water body. The dynamic pattern can be related with available resources like water, food, vegetation and natural shelter. Some specific polynomial applications are given below

(i) Hermite Polynomial:

$$H_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$$

Comparing with equation (30) we get

$$\sum_{r=0}^{[n-1]} S_{n-r} z^r = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m n!\,(2x)^{n-2m}}{m!\,(n-2m)!}$$

And n= $\frac{n}{2} + 1$

After comparing the coefficients, we get

$$S_{n-r} = \frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!} \quad (31)$$

(ii) Laguerre Polynomial:

$$L_n(x) = \sum_{r=0}^{[n]} \frac{(-1)^r n!\,(x)^r}{(n-r)!\,(r!)^2}$$

Comparing with equation (30) we get,

$$\sum_{r=0}^{[n-1]} S_{n-r} z^r = \sum_{r=0}^{[n]} \frac{(-1)^r n!(x)^r}{(n-r)!(r!)^2}$$

Where n= n+1

Then after comparing the coefficients, we get

$$S_{n-r} = \sum_{r=0}^{[n]} \frac{(-1)^r n!}{(n-r)!(r!)^2} \quad (32)$$

(iii) Jacobi Polynomials:

$$P_n^{(\alpha,\delta)}(1-\alpha) = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n\alpha^k}{n!(1+\gamma)_k 2^k}$$

We consider the case when $\alpha < 1$ i.e., when per capita birth rate is smaller than death rate.

Comparing with equation (30) we get,

$$\sum_{r=0}^{n-1} S_{n-r} z^r = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n\alpha^k}{n!(1+\gamma)_k 2^k}$$

Where n= n+1

Then after comparing the coefficients, we get

$$S_{n-r} = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n}{n!(1+\gamma)_k 2^k} \quad (33)$$

(iv) Gauss Hyper Geometric Function:

$$2F_1[-n,b;c;y] = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$$

We consider the case when $\alpha < 1$ then, from equation (30) we get,

$$\sum_{r=0}^{n-1} S_{n-r} z^r = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$$

Where n= n+1

Then after comparing the coefficients, we get

$$S_{n-r} = \frac{(-n)_r(b)_r}{(c)_r r!} \quad (34)$$

3.Numerical Examples: If$\alpha_1 = \alpha_2 = \alpha = 25/100$

$\beta_1 = \beta_2 = \beta = 0.001$

$\delta_1 = \delta_2 = \delta = \frac{50}{100}$

$\mu_1 = \mu_2 = \mu = 0.002$

$T_1 = T_2 = T = \frac{4}{100}$

$M_1 = M_2 = M = \frac{10}{100}$

$N_1 = N_2 = N = \frac{5}{100}$,   λ= 2

$P_0 = 1000$, $Q_0{}^{(2)} = 5$, $Q_0{}^{(1)} = 2$

The calculation of population figures have been done excluding the hunting or natural death of the species. This factor will significantly bring down the values of population figures as depicted in the figures. The inclusion of this factor will help us to fine-tune the population figures to acceptable values.

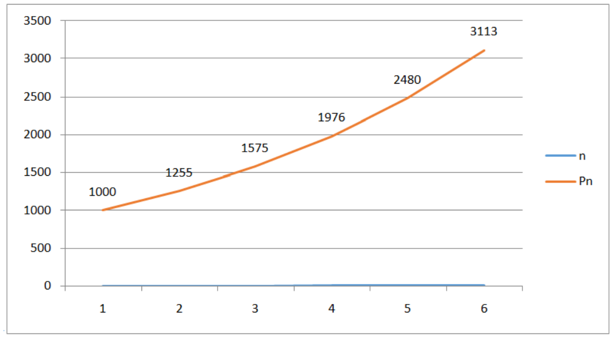

Fig. 1. Graph between $Pn^{(1)}$ and n.

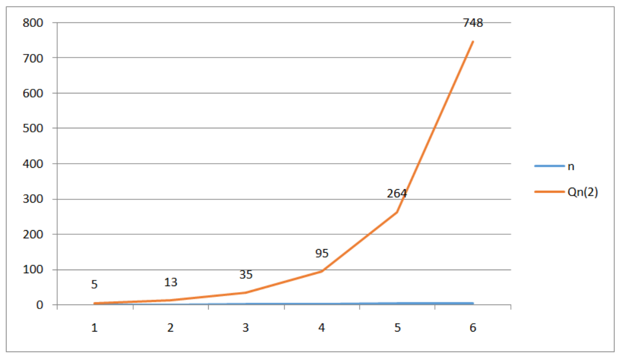

Fig. 2. Graph between $Qn^{(2)}$ and n.

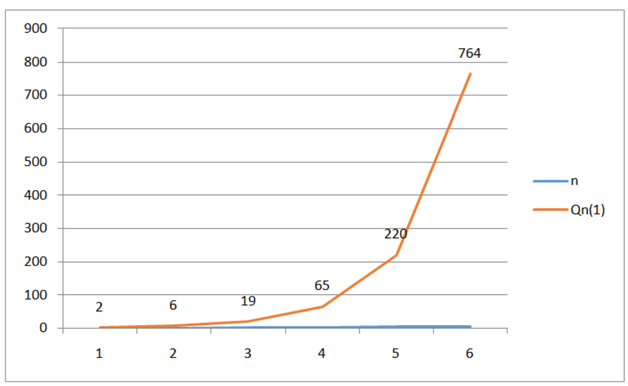

Fig. 3. Graph between $Qn^{(1)}$ and n.

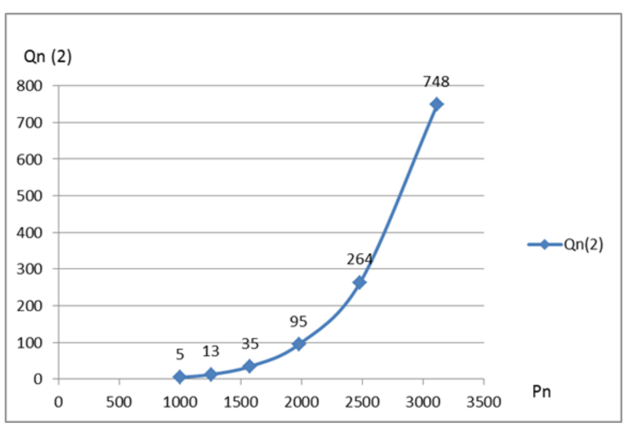

Fig. 4. Graph between $Qn^{(2)}$ Vs P_n.

III. CONCLUSIONS AND SCOPE FOR FUTURE RESEARCH

1. Our model tries to describe the population of prey and predators for different generations based on the birth rate, death rate and interaction between these two species. The model can be utilized to predict the population growth of such interacting species in a confined environment.
2. The food availability for prey population has been considered to be significantly high which shows that there is no competition among prey population for primary food sources. However, this assumption may not match with actual realities and competition for food might be a crucial factor which future research studies might like to take in to account. The future research study should focus on realistic food availability for prey population which will have a moderating effect on their growth rate of prey population as competition increases. It will subsequently result in stagnant growth rate of predator population with time.
3. Interestingly our mathematical formalism depicts strong dependence of growth of predators upon the food availability (i.e. prey population) not only for current generation but also all for previous generations. The increasing prey population across the generations creates a huge increase in the population of predators. Hence the growth of predator varies slowly in the beginning and growth rate increases across the generation.
4. The population growth rate of prey as well as predator increases slowly in the beginning but subsequently picks up. It is primarily because of the fact that food availability for the predator grows with time which results in faster effective growth rate in subsequent generation.
5. The generation dependant migration and its consequence on the evolution of single wild life species as depicted in Saxena (2011) are interestingly applicable in the Prey-Predator dynamics as well. However, the generation dependant migration is replaced by some other generation dependant terms with altogether different physical significance [14].

Our study has ignored the external factors which might affect the population dynamics of any ecosystem. Such effects have been studied in some other contexts [15]. It would be interesting to include the effect of pollution in the evolution of Prey and Predators Models. Future research studies may like to include this interesting possibility. Further the stability analysis of the prey predator dynamics which has been extensively studied in several other contexts can be included in our mathematical formalism [16].

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How to cite this article: V.P. Saxena and Lalita Dhurve (2022). Mathematical Modelling of Prey and Predator Species with Two Age Group of Prey Population. *International Journal on Emerging Technologies, 13*(2): 06–11.